Lecture 18. Methods for solving the radiative transfer equation. Part 1: Two-stream approximations.

Objectives:

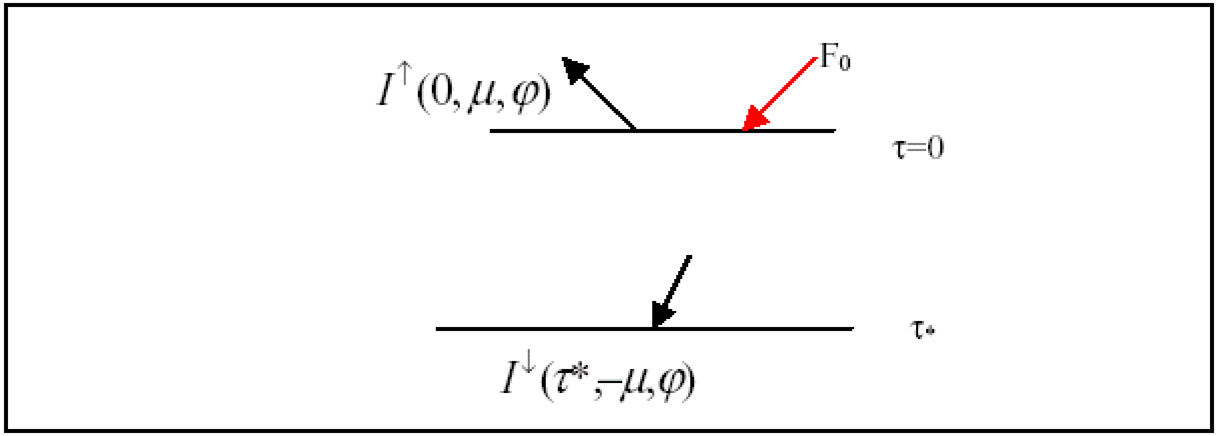
1. Concepts of the reflection and transmission of an atmospheric layer.
2. Two-stream approximations.
3. Eddington approximation.
4. Delta-function scaling.

Required reading:

L02: 6.3.1, 6.5

1. Concepts of transmission and reflection in an atmospheric layer.

Consider an atmosphere with optical depth τ_*



$I^{\uparrow}(0,\mu,\varphi)$ can be considered as the reflected diffuse intensity

$I^{\downarrow}(\tau^*,-\mu,\varphi)$ can be considered as transmitted diffuse intensity

Reflection function of an atmospheric layer is defined as

$$R(\mu,\varphi,\mu_0,\varphi_0) = \pi I^{\uparrow}(0,\mu,\varphi) / \mu_0 F_0 \qquad [18.1]$$

Transmission function of an atmospheric layer is defined as

$$T(\mu,\varphi,\mu_0,\varphi_0) = \pi I^{\downarrow}(\tau^*,-\mu,\varphi) / \mu_0 F_0 \qquad [18.2]$$

NOTE: Eq.[18.2] uses the diffuse intensity, therefore $T(\mu, \varphi, \mu_0, \varphi_0)$ is also called the diffuse transmission function.

Transmission function for direct solar radiation is defined as

$$T_{dir}(\mu_0, \varphi_0) = I_{dir}(\tau^*, -\mu_0, \varphi_0) / \mu_0 F_0 = \exp(-\tau^* / \mu_0) \qquad [18.3]$$

Planetary albedo (or local albedo or reflection) is associated with the reflected (upward) flux and defined as

$$r(\mu_0) = \frac{F_{dif}^{\uparrow}(0)}{\mu_0 F_0} = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 R(\mu, \varphi, \mu_0, \varphi_0) \mu \, d\mu \, d\varphi \qquad [18.4]$$

Diffuse transmission is associated with transmitted (downward) flux and defined as

$$t(\mu_0) = \frac{F_{dif}^{\downarrow}(\tau^*)}{\mu_0 F_0} = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 T(\mu, \varphi, \mu_0, \varphi_0) \mu \, d\mu \, d\varphi \qquad [18.5]$$

For the azimuthally independent case, Eqs.[18.4]-[18.5] reduce to

$$r(\mu_0) = 2 \int_0^1 R(\mu, \mu_0) \mu \, d\mu \qquad [18.6]$$

$$t(\mu_0) = 2 \int_0^1 T(\mu, \mu_0) \mu \, d\mu \qquad [18.7]$$

Consider a planet of radius a. The total amount of energy per unit time is

$$\pi a^2 F_0$$

Spherical (or global) albedo is a ratio of the energy reflected by the entire planet to the energy incident on it and defined as

$$\bar{r} = \frac{f^{\uparrow}(0)}{\pi a^2 F_0} = 2\int_0^1 r(\mu_0)\mu_0 d\mu_0 \qquad [18.8]$$

Global diffuse transmission is defined as

$$\bar{t} = \frac{f^{\downarrow}(\tau_1)}{\pi a^2 F_0} = 2\int_0^1 t(\mu_0)\mu_0 d\mu_0 \qquad [18.9]$$

2. Two-stream approximations.

- Two-stream methods (such as Eddington's) provide analytical solutions to the single layer plane-parallel radiative transfer equation.
- There are many related two-stream methods that approximate the angular radiance field with two numbers:
  e.g. constant hemisphere $[I^+, I^-]$, two point quadrature $[I(+\mu_1), I(-\mu_1)]$, Eddington - 0'th and 1'st moment $[I(\mu) = I_0 + I_1\mu]$.
- These methods are generally only accurate for fluxes.
  However, through a two step process, Eddington's second approximation gives accurate radiances.
- Two-stream methods are used where computational speed is important, such as climate models.

Fourier azimuth series of RTE

The plane-parallel solar radiative transfer equation is

$$\mu \frac{dI(\mu,\phi)}{d\tau} = I(\mu,\phi) - \frac{\omega}{4\pi}\int_0^{2\pi}\int_{-1}^{1} P(\Theta)\, I(\mu',\phi')\, d\mu' d\phi'$$
$$-\frac{\omega}{4\pi} P(\mu,\phi;-\mu_0,\phi_0+\pi)\, S_0 e^{-\tau/\mu_0}$$

where the last term is the pseudosource of diffuse radiation.

Plane-parallel radiative transfer is often solved with a Fourier series in ϕ:

$$I(\tau, \mu, \phi) = \sum_{m=0}^{N} I_m(\tau, \mu) \cos m(\phi_0 - \phi)$$

The $m = 0$ term is the azimuthal average, $I_0(\tau, \mu)$.

Use addition theorem of spherical harmonics for phase function

$$P(\mu, \phi; \mu', \phi') = \sum_{m=0}^{N} \sum_{l=m}^{N} \omega_l \, a_{lm} \, P_l^m(\mu) P_l^m(\mu') \cos m(\phi' - \phi)$$

where $a_{lm} = (2 - \delta_{0,m}) \frac{(l-m)!}{(l+m)!}$ and the Legendre series coefficients ω_l are defined by $P(\cos\Theta) = \sum_{l=0}^{N} \omega_l \mathcal{P}_l(\cos\Theta)$.

This is the major reason for using Legendre series for phase functions. Substitute the Fourier series for $I(\mu, \phi)$ and the addition theorem phase function in RTE. Scattering integral has $\int \cos m(\phi - \phi') \cos m'(\phi - \phi') d\phi = \delta_{mm'}$ which gets rid of m sum.

Radiative transfer equation becomes (leaving off diffuse source)

$$\mu \frac{dI_m(\tau, \mu)}{d\tau} = I_m(\tau, \mu) - \frac{\omega}{2} \sum_{l=m}^{N} a_{lm} \omega_l P_l^m(\mu) \int_{-1}^{+1} P_l^m(\mu') I_m(\tau, \mu') d\mu'$$

Fourier azimuthal modes separate: N+1 separate equations ($m = 0, \ldots, N$).

Simple approximation for intensity.

Underlying idea:

Because radiation flux and heating rates are angular-averaged properties, one can expect that details of the angular variation of intensity are not very important for the predictions of these quantities.

Strategy:

Introduce an "effective" angular averaged intensity (stream). But one must decide on how to determine the "effective" intensity (i.e., the effective scattering angle $\bar{\mu}^{\uparrow\downarrow}$).

Disadvantages of the two-stream approximations:

Two-stream methods provide acceptable accuracy but over a restricted range of the parameters. There is no a priory method to estimate the accuracy, so one needs to use the "exact" method to obtain an accurate solution which can be used to estimate the accuracy of two-stream solutions.

Advantages of the two-stream approximations:

Two-stream approximations are computationally efficient (therefore they are often used in climate models) and often sufficiently accurate.

Azimuthally Averaged RTE

Fluxes need only $m = 0$ mode:

$$F^{\uparrow}(\tau) = \int_0^{2\pi} \int_0^1 I(\tau, \mu)\, \mu\, d\mu\, d\phi \;=2\pi \int_0^1 I_{m=0}(\tau, \mu) \mu\, d\mu$$

Azimuthally averaged phase function

$$P(\mu, \mu') = \frac{1}{2\pi} \int_0^{2\pi} P(\mu, \phi; \mu', \phi') d\phi' \;= \sum_{l=0}^{N} \omega_l\, P_l(\mu) P_l(\mu')$$

Possible strategies to define the effective scattering angle:

i) define $\overline{\mu}^{\uparrow\downarrow}$ as the intensity-weighted angular means

$$\overline{\mu}^{\uparrow\downarrow} = \frac{\int_0^1 I^{\uparrow\downarrow}(\tau, \mu) \mu\, d\mu}{\int_0^1 I^{\uparrow\downarrow}(\tau, \mu)\, d\mu} \qquad [18.10]$$

For **isotropic radiation filed**, Eq.[18.10] gives $\overline{\mu}^{\uparrow\downarrow} = 1/2$

ii) define $\overline{\mu}^{\uparrow\downarrow}$ as the root-mean square value

$$\overline{\mu}^{\uparrow\downarrow} = \sqrt{\langle \mu^2 \rangle} = \sqrt{\frac{\int_0^1 I^{\uparrow\downarrow}(\tau, \mu) \mu^2\, d\mu}{\int_0^1 I^{\uparrow\downarrow}(\tau, \mu)\, d\mu}} \qquad [18.11]$$

For **isotropic radiation filed**, Eq.[18.11] gives $\overline{\mu}^{\uparrow\downarrow} = 1/\sqrt{3}$

NOTE: A problem with the above two approaches (Eqs.[18.10] and [18.11]) is that we don't know a priori the angular distribution on the intensity.

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A better strategy is to utilize the Gaussian quadratures

Gaussian quadrature applied to any function $\boldsymbol{f(\mu)}$ gives

$$\int_{-1}^{1} f(\mu)d\mu \approx \sum_{j=-n}^{n} a_j f(\mu_j) \qquad [18.12]$$

where $\boldsymbol{a_j}$ are the weights defined as

$$a_j = \frac{1}{P_{2n}'(\mu_j)} \int_{-1}^{1} \frac{P_{2n}(\mu)}{\mu - \mu_j} d\mu \qquad [18.13]$$

and μ_j are the zeros of the even-order Legendre polynomials $\boldsymbol{P_{2n}(\mu)}$, and the prime denotes the differentiation with respect to μ_j.

NOTE: Table 6.1 in L02 lists Gaussian points μ_j and weights $\boldsymbol{a_j}$ for n = 1, 2, 3 and 4.

Recall the equation of the radiative transfer for the diffuse intensity Eq.[17.7] for the azimuth-independent case

$$\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{\omega_0}{2}\sum_{l=0}^{N} \varpi_l^* P_l(\mu)\int_{-1}^{1} P_l(\mu')I(\tau,\mu')d\mu' -$$

$$-\frac{\omega_0}{4\pi}\sum_{l=0}^{N} \varpi_l^* P_l(\mu)P_l(-\mu_o)F_0 \exp(-\tau/\mu_0)$$

Using Gaussian quadratures, we can re-write this equation as

$$\mu_i \frac{dI(\tau,\mu_i)}{d\tau} = I(\tau,\mu_i) - \frac{\omega_0}{2}\sum_{l=0}^{N} \varpi_l^* P_l(\mu_i)\sum_{j=-n}^{n} a_j P_l(\mu_j)I(\tau,\mu_j) -$$

$$-\frac{\omega_0}{4\pi}[\sum_{l=0}^{N} (-1)^l \varpi_l^* P_l(\mu_i)P_l(-\mu_o)] F_0 \exp(-\tau/\mu_0)$$

[18.14]

where ***i* = -*n*, *n*** and **μ_i(*-n, n)*** represent the directions of radiation streams.

In the two-stream approximation, one takes only two streams (i.e., j =-1 and 1) and N=1.

Note in table 6.1 in L02 that $\mu_1 = \dfrac{1}{\sqrt{3}}$ and $\boldsymbol{a_j} = \boldsymbol{a_{-j}} = 1$

For this case, Eq.[18.14] splits into two equations

$$\mu_1 \frac{dI^{\uparrow}(\tau,\mu_1)}{d\tau} = I^{\uparrow}(\tau,\mu_1) - \omega_0(1-b)I^{\uparrow}(\tau,\mu_1) - \omega_0 b I^{\downarrow}(\tau,-\mu_1) - S^{-}\exp(-\tau/\mu_0)$$

[18.15a]

$$-\mu_1 \frac{dI^{\downarrow}(\tau,-\mu_1)}{d\tau} = I^{\downarrow}(\tau,-\mu_1) - \omega_0(1-b)I^{\downarrow}(\tau,-\mu_1) - \omega_0 b I^{\uparrow}(\tau,\mu_1) - S^{+}\exp(-\tau/\mu_0)$$

[18.15b]

where

$$S^{\pm} = \frac{F_0\omega_0}{4}(1 \pm 3g\mu_1\mu_0)$$

$$g = \frac{\varpi_1^*}{3} = \frac{1}{2}\int_{-1}^{1} P(\cos\Theta)\cos(\Theta)\,d\cos(\Theta)\,,$$ ***g*** is the asymmetry parameter.

$$b = \frac{1-g}{2} = \frac{1}{2}\int_{-1}^{1} P(\cos\Theta)\frac{1-\cos(\Theta)}{2}\,d\cos(\Theta)\,;$$ ***b*** can be interpreted as a backscattered fraction of energy and ***(1-b)*** is forward scattered energy.

The solutions of Eqs.[18.15a,b] are (see L02,pp.305-306)

$$\boxed{\begin{aligned} I^{\uparrow} &= I(\tau,\mu_1) = Kv\,\exp(k\tau) + Hu\,\exp(-k\tau) + \varepsilon\,\exp(-\tau/\mu_0) \quad &[18.16a] \\ I^{\downarrow} &= I(\tau,-\mu_1) = Ku\,\exp(k\tau) + Hv\,\exp(-k\tau) + \gamma\,\exp(-\tau/\mu_0) \quad &[18.16b] \end{aligned}}$$

where

$$v = (1+a)/2; \quad u = (1-a)/2; \; a^2 = \frac{1-\omega_0}{1-g\omega_0}; \quad k^2 = \frac{(1-\omega_0)(1-g\omega_0)}{\mu_1^2}$$

$$\varepsilon = \frac{\alpha+\beta}{2}; \quad \gamma = \frac{\alpha-\beta}{2}; \quad \alpha = \frac{Z_1\mu_0^2}{1-\mu_0^2k^2}; \quad \beta = \frac{Z_2\mu_0^2}{1-\mu_0^2k^2}$$

$$Z_1 = -\frac{(1-g\omega_0)(S^- + S^+)}{\mu_1^2} + \frac{S^- + S^+}{\mu_1\mu_0}; \quad Z_2 = -\frac{(1-\omega_0)(S^- - S^+)}{\mu_1^2} + \frac{S^- + S^+}{\mu_1\mu_0}$$

The constant ***K*** and ***H*** are determined from the boundary conditions on the top and at the bottom of the atmospheric layer. Using the boundary conditions given by Eq.[17.10] (i.e., no diffuse downward radiation at the top of the atmosphere and no reflection from the surface), we have

$$K = -\frac{\varepsilon v \exp(-\tau^* / \mu_0) - \gamma u \exp(-k\tau^*)}{v^2 \exp(k\tau^*) - u^2 \exp(-k\tau^*)}$$

$$H = -\frac{\varepsilon u \exp(-\tau^* / \mu_0) - \gamma v \exp(-k\tau^*)}{v^2 \exp(k\tau^*) - u^2 \exp(-k\tau^*)}$$

From the upward and downward intensities we can write expressions for **upward and downward diffuse fluxes** in the two-stream approximations:

$$\boxed{F^{\uparrow}(\tau) = 2\pi\mu_1 I^{\uparrow}(\tau, \mu_1)} \qquad [18.17a]$$

$$\boxed{F^{\downarrow}(\tau) = 2\pi\mu_1 I^{\downarrow}(\tau, -\mu_1)} \qquad [18.17b]$$

3. Eddington approximation.

Azimuthally averaged diffuse intensity and scattering phase function may be expanded in terms of Legendre polynomials as

$$I(\tau,\mu) = \sum_{l=0}^{N} I_l(\tau) P_l(\mu)$$

$$P(\mu,\mu') = \sum_{l=0}^{N} \varpi_l^* P_l(\mu) P_l(\mu')$$

Note that $P_0(\mu) = 1$ and $P_1(\mu) = \mu$.

Strategy of the Eddington approximation:

Approximate the radiance field and scattering phase function to first order in μ.

From the above equations,

$$I(\tau,\mu) = I_0(\tau) + I_1(\tau)\mu \;; \quad -1 \le \mu \le 1 \qquad [18.18]$$

$$P(\mu,\mu') = 1 + 3g\mu\mu' \qquad [18.19]$$

Put Eqs.[18.18] and [18.19] into the azimuthally averaged radiative transfer equation (Eq.[17. 7), we have

$$\mu\frac{d(I_0+I_1\mu)}{d\tau}=(I_0+I_1\mu)-\frac{\omega_0}{2}\int_{-1}^{1}(I_0+I_1\mu)(1+3g\mu\mu')d\mu'-\frac{\omega_0}{4\pi}F_0(1-3g\mu\mu_0)\exp(-\tau/\mu_0) \qquad [18.20]$$

Doing the integral in Eq.[18.20] results in

$$\mu\frac{d(I_0+I_1\mu)}{d\tau}=(I_0+I_1\mu)-\omega_0(I_0+I_1g\mu)-\frac{\omega_0}{4\pi}F_0(1-3g\mu\mu_0)\exp(-\tau/\mu_0) \qquad [18.21]$$

Rearranging terms gives

$$\mu\frac{dI_0}{d\tau}+\mu^2\frac{dI_1}{d\tau}=I_0(1-\omega_0)+I_1(1-g\omega_0)\mu-\frac{\omega_0}{4\pi}F_0(1-3g\mu\mu_0)\exp(-\tau/\mu_0) \qquad [18.22]$$

First integrate over μ from –1 to 1 and then multiply by μ and integrate from –1 to 1 to get two coupled equations:

$$\frac{dI_1}{d\tau}=3(1-\omega_0)I_0-\frac{3}{4\pi}\omega_0F_0\exp(-\tau/\mu_o) \qquad [18.23a]$$

$$\frac{dI_0}{d\tau}=3(1-\omega_0g)I_1+\frac{3}{4\pi}\omega_0g\mu_0F_0\exp(-\tau/\mu_o) \qquad [18.23b]$$

Differentiate Eq.[18.23b] by τ and substitute in Eq.[18.23a]

$$\frac{d^2 I_0}{d\tau^2} = k^2 I_0 - \frac{3}{4\pi}\omega_0 (1 + g - g\omega_0) F_0 \exp(-\tau/\mu_o) \qquad [18.24]$$

where $k^2 = \frac{(1-\omega_0)(1-g\omega_0)}{\mu_1^2}$ is the eigenvalue.

NOTE: Eq.[18.24] is known as diffusion equation for radiative transfer.

The solution of Eq.[18.24] for $\boldsymbol{I_o}$ is exponential in τ:

$$\boxed{I_0 = K \exp(k\tau) + H \exp(-k\tau) + \Psi \exp(-\tau/\mu_o)} \qquad [18.25]$$

where

$$\Psi = \frac{3}{4\pi}\omega_0 F_0 \frac{1 + g(1-\omega_0)}{k^2 - 1/\mu_o^2}$$

and the integration constants $\boldsymbol{K}$ and $\boldsymbol{H}$ are to be determined from the boundary conditions.

Similarly, the solution for $\boldsymbol{I_1}$ can be determined as

$$I_1 = aK \exp(k\tau) - aH \exp(-k\tau) - \xi \exp(-\tau/\mu_o) \qquad [18.26]$$

where $\boldsymbol{a^2 = 3(1-\omega_0)(1-\omega_0 g)}$

$$\xi = \frac{3}{4\pi}\omega_0 \frac{F_0}{\mu_0} \frac{1 + 3g(1-\omega_0)\mu_0^2}{k^2 - 1/\mu_o^2}$$

Thus the diffuse fluxes in the Eddington approximation are

$$F^{\uparrow}(\tau) = 2\pi \int_{0}^{1} [I_0(\tau) + \mu I_1(\tau)]\mu \, d\mu = \pi\left[I_0(\tau) + \frac{2}{3} I_1(\tau)\right] \qquad [18.27a]$$

$$F^{\downarrow}(\tau) = 2\pi \int_{0}^{-1} [I_0(\tau) + \mu I_1(\tau)]\mu \, d\mu = \pi\left[I_0(\tau) - \frac{2}{3} I_1(\tau)\right] \qquad [18.27b]$$

- The two-stream and Eddington methods are good approximations for optically thick layer, but they may produce inaccurate results for thin layers and strong absorption. The main problem is that the phase function is highly peaked in the forward direction.

For the optically thin atmosphere, the albedo and diffuse transmission are

$$r(\mu_0) = \omega_0 (1/2 - 3 g \mu_0 / 4)\tau^* / \mu_o \qquad [18.28]$$

$$t(\mu_0) = 1 - r - (1 - \omega_0)\tau^* / \mu_o \qquad [18.29]$$

Problem: negative reflected flux for $g\mu_0 > 2/3$

Eddington Solution Results

The standard boundary conditions for a single layer are no incident radiation from above and below: $F^{\downarrow}(0) = 0$ and $F^{\uparrow}(\tau^*) = 0$.
The solution is quite complicated (e.g. see Meador and Weaver, 1980), so we look at two special cases for a uniform layer of optical depth τ^*.

The optically thin solution for reflected and transmitted flux fraction is

$$R = \omega(1/2 - 3g\mu_0/4)\tau^*/\mu_0 \qquad T = 1 - R - (\tau^*/\mu_0)(1-\omega)$$

Problem: negative reflected flux for $g\mu_0 > 2/3$.

- ***Example of Eddington solution results***

Consider a uniform layer of optical depth τ^*. In the Eddington approximation for conservative scattering (ω_o=1), the **albedo** (fractional reflected flux) of the layer is

$$\boxed{r(\mu_0) = \frac{(1-g)\tau^* + (2/3 - \mu_0)(1 - \exp(-\tau^*/\mu_0))}{4/3 + (1-g)\tau^*}} \qquad [18.30]$$

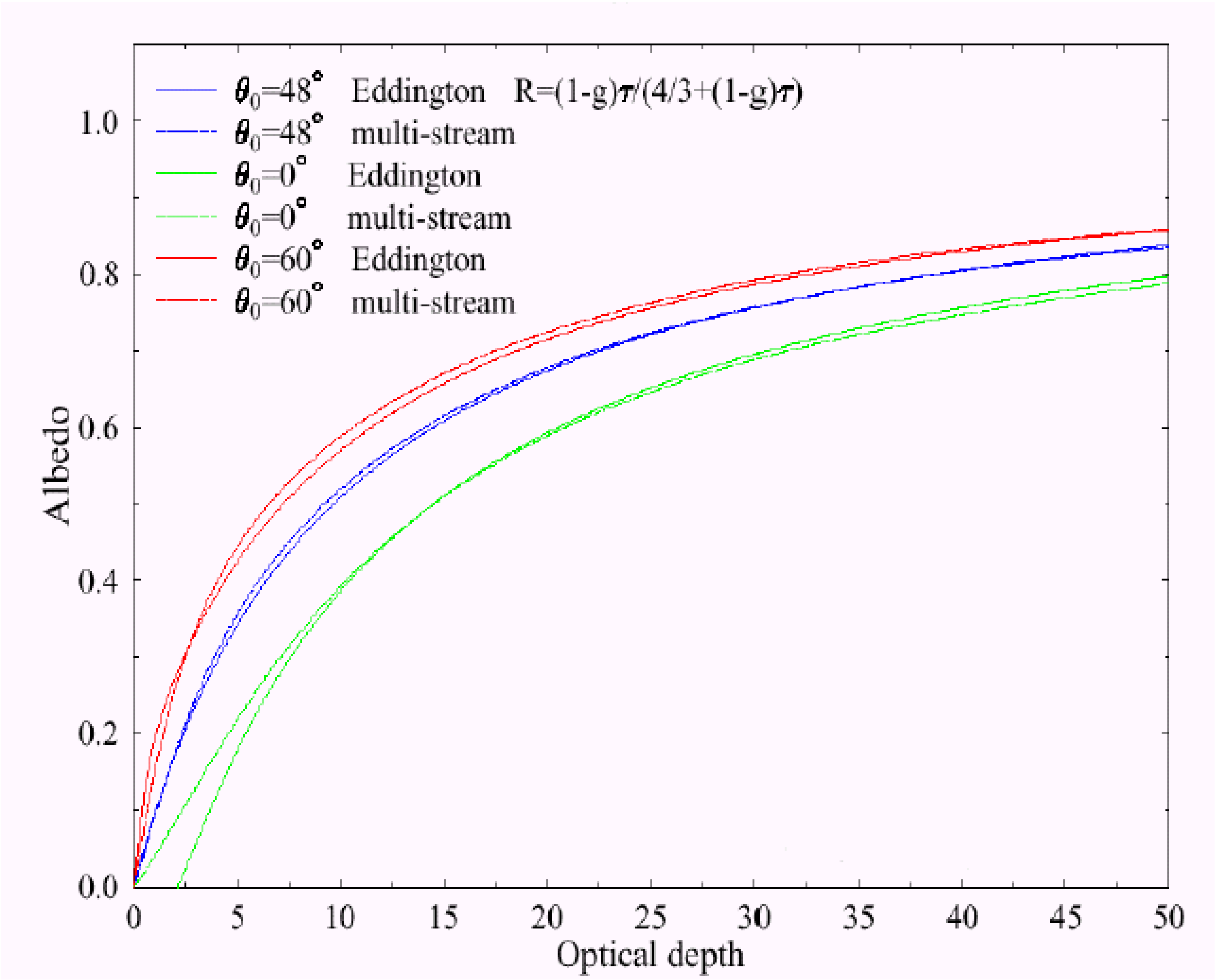


Figure 18.1 Comparison of Eddington and multistream albedo for conservative scattering ($\lambda = 0.65$ µm, $r_e = 10$ µm, $g = 0.862$, i.e. cloud albedo)

Some properties of reflectivity:

- ✓ Linear for τ <<1 and saturation for τ >>1
- ✓ More forward scattering means less reflection ($g\ \uparrow \Rightarrow\ r\ \downarrow$)
- ✓ Higher solar zenith angle means more reflection unless optically thin ($\mu_0 \downarrow \Rightarrow r \uparrow$)
- ✓ Multiple scattering amplifies absorption (μ_o= 2/3; g =0.85)

τ =1, ω_o= 0.99 => r = 0.096, A = 0.018

τ =10, ω_o= 0.99 => r = 0.45, A = 0.18

τ =100, ω_o= 0.99 => r = 0.55, A = 0.45

τ =10, ω_o= 1 => r = 0.92, A = 0.

4. Delta-function scaling.

Scattering by atmospheric particulates has the forward diffraction peak and therefore two-term expansion of the scattering phase function (as was done above) is not adequate.

Delta-function adjustment replaces a highly peaked phase function with:

(1) a delta function in the forward direction

(2) a smoother scaled phase function (P')

Delta scaling of phase function with **forward scattering fraction f**:

$$P(\cos\Theta) = 2f\delta(1-\cos\Theta) + (1-f)P'(\cos\Theta) \qquad [18.31]$$

Thus the asymmetry parameter is

$$g = \frac{1}{2}\int_{-1}^{1} P(\cos\Theta)\cos\Theta\, d\cos\Theta = f + (1-f)g' \qquad [18.32]$$

where g' is the scaled asymmetry parameter.

The scaled scattering and absorption optical depth must be

$$\tau_s' = (1-f)\tau_s \quad \text{and} \quad \tau_a' = \tau_a$$

Delta Scaling of the Radiative Transfer Equation

Can delta scale any form of the radiative transfer equation.

Azimuthally averaged RTE:

$$\mu\frac{dI}{d\tau} = I(\tau,\mu) - \frac{\omega}{2}\int_{-1}^{1} P(\mu,\mu')I(\tau,\mu')d\mu' + S(\tau,\mu)$$

Put in our approximation for the true phase function

$$\mu\frac{dI}{d\tau} = I - f\omega I - \frac{1}{2}(1-f)\omega\int_{-1}^{1} P'(\mu,\mu')I(\mu')d\mu' + S'$$

$$\mu\frac{dI}{(1-\omega f)d\tau} = I - \frac{1}{2}\frac{(1-f)\omega}{1-\omega f}\int_{-1}^{1} P'(\mu,\mu')I(\mu')d\mu' + S'$$

This is the same radiative transfer equation if we scale variables!

$$\tau' = (1-\omega f)\tau \quad \omega' = \frac{(1-f)\omega}{1-\omega f} \quad g' = \frac{g-f}{1-f}$$

Procedure: delta scale extinction ($\beta' = (1-\omega f)\beta$), single scattering albedo, and phase function, then use in regular radiative transfer equation.

How does scaling change the optical properties?

- **Delta-function adjustment** is introduced to incorporate the forward peak contribution by adjusting optical properties such that the fraction of scattered energy in the forward direction, **f,** is removed from the scattering parameters

$$g' = \frac{g - f}{1 - f} \qquad \omega_o' = \frac{(1 - f)\omega_0}{1 - f\omega_0} \qquad \tau' = (1 - f\omega_0)\tau$$

- The incorporation of the δ-function adjustment into two-stream and Eddington methods greatly improves their accuracy.

<u>How to get the delta scaling fraction f</u>

Delta-isotropic: make scaled phase function isotropic g'= 0 => f = g

Delta-Eddington: make two term scaled function: choose $f = \frac{\varpi_2}{5}$

For instance, for Henyey-Greenstein phase function: f = g^2

thus $g' = \frac{g}{1 + g}$; $\omega_o' = \frac{(1 - g^2)\omega_0}{1 - g^2\omega_0}$; $\tau' = (1 - g^2\omega_0)\tau$

Delta-M Approximation: gives accurate fluxes in numerical radiative transfer models with M discrete "streams" per hemisphere:

$$\chi_l' = \frac{\chi_l - f}{1 - f} \quad l < 2M \quad f = \chi_{2M} \qquad \chi_l = \frac{\omega_l}{2l+1}$$

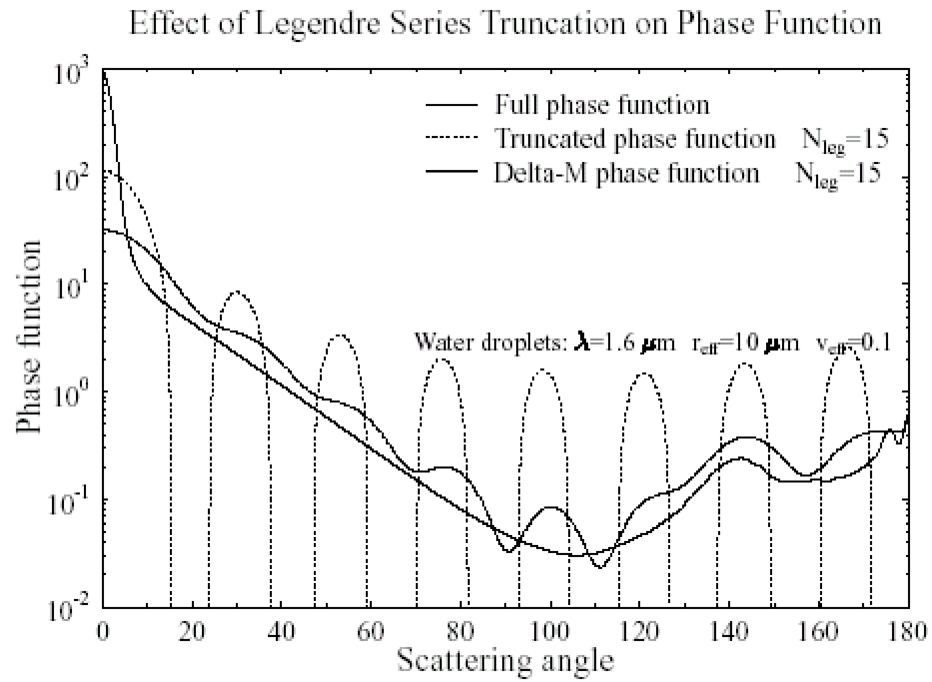
Delta-Eddington is case with $M = 1$.
Delta-M phase function has much less oscillation than truncated phase function ($\omega_l = 0$ for $l > 2M$).

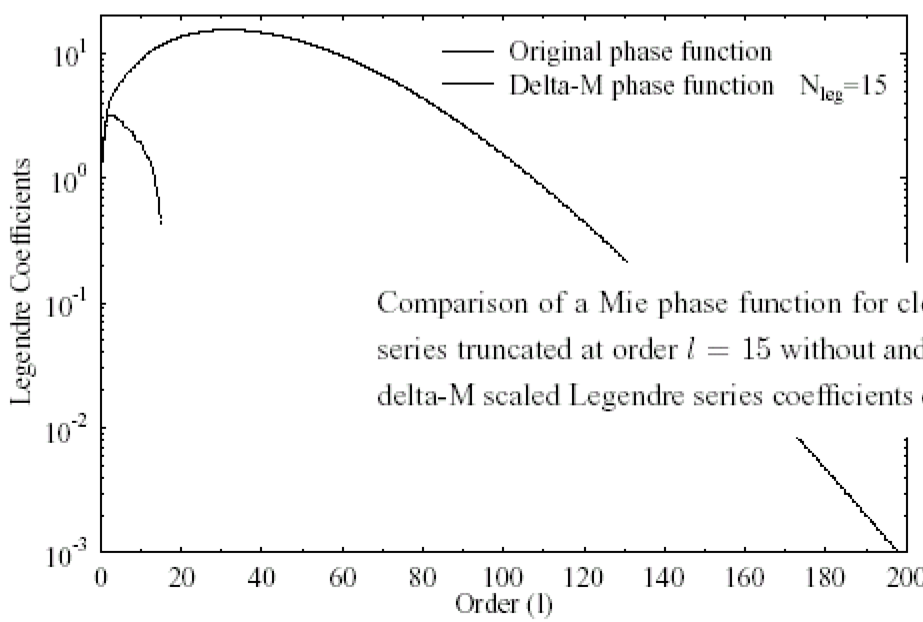
Delta Scaling Summary

Scaled radiative transfer system has:
Lower optical depth + less forward scattering = same reflection, etc.

Similarity principle: τ, ω, g is equivalent to τ', ω', g'

Lower scaled optical depth $\rightarrow$ higher direct beam transmission.
Must add diffuse and direct transmitted flux to get "correct" total.





Comparison of a Mie phase function for cloud droplets with the phase function of the Legendre series truncated at order $l = 15$ without and with delta-M scaling applied (top). The original and delta-M scaled Legendre series coefficients of the phase function (bottom).